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HEAT TRANSFER ACROSS A TURBULENT BOUNDARY LAYER: APPLICATION OF A PROFILE METHOD TO THE STEP-WALL-TEMPERATURE PROBLEM

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NOMENCLATURE*

a, a constant;
b, exponent in the temperature profile;
c_f, local drag coefficient;
E, a constant;
K, a constant;
N_{Pr}, laminar Prandtl number;
$N_{Pr,t}$, turbulent Prandtl number;
N_{St}, Stanton number;
S_T, non-dimensional heat-transfer coefficient (12);
T, temperature;
u^+, non-dimensional velocity along the wall;
x^+, non-dimensional distance along the wall;
X, non-dimensional distance (5);
y^+, non-dimensional distance normal to the wall;
δ, non-dimensional thickness of the temperature profile;
θ, non-dimensional temperature (10);
ξ, non-dimensional distance (4);
ϕ, turbulent contribution to the non-dimensional total viscosity.

Subscripts

G, main stream;
S, wall.

1. INTRODUCTION

FOLLOWING a method put forward by Spalding [1], Gardner and Kestin [2] obtained exact numerical solutions of the partial differential equation for heat transfer through uniform-property "universal" turbulent boundary layer, for various Prandtl numbers. In a later paper [3], Spalding generalized the solutions in reference [2] for non-unity turbulent Prandtl number and also gave exact analytical solutions for the case of $N_{Pr} = N_{Pr,t}$ with a power-law velocity profile.

* Nomenclature is as in reference [3] where further details can be found. The numbers in parentheses denote the defining equations.

It is the purpose of the present communication to present a profile method for solving the same problem and to examine the extent to which the resulting approximate solutions agree with the exact solutions. The special feature of this method is that two integral equations are used in conjunction with a power-law temperature profile, the two free parameters in the profile being the exponent and the "thickness" of the profile.

The step-wall-temperature problem of a turbulent boundary layer is taken here to illustrate the method; however, the method can also be applied to many problems of the boundary-layer type in fluid mechanics, heat conduction and diffusion.

2. DESCRIPTION OF THE METHOD

Here all the equations will be given in the generalized form according to reference [3].

2.1 *The partial differential equation*

The temperature T in a uniform-property universal turbulent boundary layer has been shown in reference [3] to be governed by:

$$\frac{\partial T}{\partial(x^+/N_{Pr,t})} = \frac{1}{u^+(1+\phi)}\frac{\partial}{\partial u^+} \times \left\{\frac{N_{Pr,t}/N_{Pr} + \phi}{1+\phi}\frac{\partial T}{\partial u^+}\right\}. \quad (1)$$

Here ϕ is to be obtained from the universal velocity profile by the relation:

$$1 + \phi = \frac{dy^+}{du^+}. \quad (2)$$

Equation (1) represents the conservation of enthalpy in the boundary layer for uniform values of density and specific heat (i.e. small velocities and temperature differences). The two fundamental assumptions made in obtaining this equation are: that there is a universal relationship between u^+ and y^+; and that the shear stress is uniform across the boundary layer. These assumptions are known to hold quite well, at least in the part of the boundary layer nearer to the wall and, therefore, equation (1) can be expected to give correct predictions of heat transfer whenever the thermal boundary layer is appreciably thinner than the velocity boundary layer.

Equation (1) can be rewritten in a more convenient form, in terms of a new independent variable ξ, as:

$$\frac{\partial T}{\partial X} = \frac{1}{u^+(N_{Pr,t}/N_{Pr} + \phi)}\frac{\partial^2 T}{\partial \xi^2}; \tag{3}$$

where

$$\xi \equiv \int_0^{u^+} \frac{1 + \phi}{(N_{Pr,t}/N_{Pr} + \phi)} du^+, \tag{4}$$

and

$$X \equiv x^+/N_{Pr,t}. \tag{5}$$

Here we consider the step-wall-temperature problem; so the boundary conditions are:

$$\left.\begin{array}{ll} X = 0, & \xi \geq 0 \\ \text{all } X, & \xi \to \infty \end{array}\right\}: \quad T = T_G; \tag{6}$$

$$X > 0, \quad \xi = 0: \quad T = T_S. \tag{7}$$

2.2 *Integral equations*

From the partial differential equation (3), we form two integral equations by integration with respect to ξ from 0 to ∞, after multiplication respectively by unity and T, as weighting functions. The resulting integral equations are:

$$\frac{d}{dX}\int_0^\infty u^+(N_{Pr,t}/N_{Pr} + \phi)\,\theta\, d\xi = -\left(\frac{\partial\theta}{\partial\xi}\right)_S; \tag{8}$$

$$\frac{1}{2}\frac{d}{dX}\int_0^\infty u^+(N_{Pr,t}/N_{Pr} + \phi)\,\theta^2\, d\xi = -\left(\frac{\partial\theta}{\partial\xi}\right)_S - \int_0^\infty \left(\frac{\partial\theta}{\partial\xi}\right)^2 d\xi; \tag{9}$$

where θ is the dimensionless temperature defined by:

$$\theta \equiv \frac{T - T_G}{T_S - T_G}. \tag{10}$$

In order to solve these integral equations we need an assumption for the temperature profile.

2.3 *The temperature profile*

We shall use the following two-parameter temperature profile.

$$\left.\begin{array}{ll} \xi \leq \delta: & \theta = (1 - \xi/\delta)^b, \\ \xi > \delta: & \theta = 0. \end{array}\right\} \tag{11}$$

Here the two parameters are: b, the exponent; and δ which has the significance of the "thickness" of the temperature profile.

The dimensionless heat-transfer coefficient, S_T, defined by:

$$S_T \equiv \frac{N_{St}\, N_{Pr}}{(c_f/2)^{\frac{1}{2}}}, \tag{12}$$

can be shown to be equal to $-(N_{Pr}/N_{Pr,t})(\partial\theta/\partial\xi)_S$. Therefore, in terms of the profile parameters, we obtain:

$$S_T = \left(\frac{N_{Pr}}{N_{Pr,t}}\right)\cdot\frac{b}{\delta}. \tag{13}$$

3. SOLUTION OF THE INTEGRAL EQUATIONS

Here we apply the method outlined above to the following three cases:

(i) laminar velocity profile: $y^+ = u^+$; $N_{Pr} = N_{Pr,t}$;
(ii) seventh-power-law velocity profile: $y^+ = au^{+7}$; $N_{Pr} = N_{Pr,t}$; and
(iii) velocity profile given by the universal law of the wall due to Spalding [4]; $N_{Pr}/N_{Pr,t}$ = 0·71, 1, 7, 30, 100, 1000.

Exact analytical solutions for the first two cases are given in reference [3]; for the third case, exact numerical solutions are presented in reference [2].

It can be easily seen that, when $N_{Pr} = N_{Pr,t}$, the new independent variable ξ becomes the same as u^+.

3.1 *Laminar velocity profile*: $y^+ = u^+$; $N_{Pr} = N_{Pr,t}$

Here

$$1 + \phi = \frac{dy^+}{du^+} = 1. \tag{14}$$

Substituting the expression for the temperature profile and eliminating δ from the two integral equations, namely (8) and (9), we get:

$$6b^2 - 7b - 2 = 0. \tag{15}$$

This equation has the positive root b = 1·404; this is the relevant one in the present problem. Integration of equation (8) then yields:

$$\delta = \{1{\cdot}5b(b + 1)(b + 2)\}^{\frac{1}{3}}\, X^{\frac{1}{3}}; \tag{16}$$

and

$$S_T = b/\delta = b^{\frac{2}{3}}\{1{\cdot}5(b + 1)(b + 2)\}^{-\frac{1}{3}}\, X^{-\frac{1}{3}}. \tag{17}$$

Substituting the value of b, we get, as the final result:

$$S_T = 0{\cdot}543X^{-\frac{1}{3}}. \tag{18}$$

The corresponding exact solution given in reference [3] is

$$S_T = 0{\cdot}53835X^{-\frac{1}{3}}. \tag{19}$$

The agreement is very satisfactory.

3.2 *Seventh-power-law velocity profile*: $y^+ = au^{+7}$; $N_{Pr} = N_{Pr,t}$

Here

$$1 + \phi = \frac{dy^+}{du^+} = 7au^{+6}. \tag{20}$$

Eliminating δ from the two integral equations, we get:

$$\frac{(2b + 1)(2b + 2)(2b + 3)\dots(2b + 8)}{(b + 1)(b + 2)(b + 3)\dots(b + 8)} = \frac{2b - 1}{2(b - 1)}. \tag{21}$$

The relevant root of this equation is $b = 1{\cdot}11$. Now the integration of (8) yields:

$$\delta = \left\{\frac{9b(b + 1)(b + 2)(b + 3)\dots(b + 8)}{56a(7!)}\right\}^{1/9} X^{1/9}; \tag{22}$$

and

$$S_T = b/\delta =$$

$$b^{8/9}\left\{\frac{56a(7!)}{9(b + 1)(b + 2)(b + 3)\dots(b + 8)}\right\}^{1/9} X^{-1/9}. \tag{23}$$

Substituting the value of b as obtained above, and taking $a = 2{\cdot}412 \times 10^{-7}$ as used in reference [3], we get, as the final solution:

$$S_T = 0{\cdot}1504X^{-1/9}. \tag{24}$$

The exact solution for this case, as given in reference [3], is

$$S_T = 0{\cdot}1479X^{-1/9}. \tag{25}$$

Again, the coefficients agree within 2 per cent.

3.3 *Universal law of the wall, various Prandtl numbers*

For this case, we use the following law of the wall due to Spalding [4]:

$$y^+ = u^+ + \frac{1}{E}\left\{e^{Ku^+} - 1 - Ku^+ - \left(\frac{Ku^+}{2!}\right)^2 - \left(\frac{Ku^+}{3!}\right)^3 - \left(\frac{Ku^+}{4!}\right)^4\right\}, \tag{26}$$

with $K = 0{\cdot}4$, and $E = 9{\cdot}025$.

This gives:

$$1 + \phi = \frac{dy^+}{du^+} = 1 + \frac{K}{E}\left\{e^{Ku^+} - 1 - Ku^+ - \frac{(Ku^+)^2}{2!} - \frac{(Ku^+)^3}{3!}\right\} \tag{27}$$

In this case, it can be seen that the exponent b does not have a constant value, but changes with X. Therefore, the two integral equations should be treated as two simultaneous ordinary differential equations with b and δ as the dependent variables. Then these can be solved by the standard techniques of numerical forward integration.

Unlike the first two cases, here it is necessary to have u^+ as a function of ξ and $N_{Pr}/N_{Pr,t}$. In the present work, this function was obtained by evaluating the quadrature in equation (4) numerically.

The solutions were obtained from $X = 1$ to $X = 10^6$ for $N_{Pr}/N_{Pr,t} = 0{\cdot}71$, 1, 7, 30, 100 and 1000. For the starting values at $X = 1$, the results of the first case (described in Section 3.1 above) were used. Thus the values of b and δ at the start of the integration were given by:

$$b = 1{\cdot}404, \tag{28}$$

and

$$\delta = \{1{\cdot}5b(b + 1)(b + 2)\}^{1/3}X^{1/3}(N_{Pr}/N_{Pr,t})^{1/3} \tag{29}$$

[This is a modified form of equation (16); the modification accounts for the fact that here $N_{Pr} \neq N_{Pr,t}$.] Since the law of the wall given by equation (26) has $y^+ = u^+$ as the asymptote for small u^+, the results of the first case form a good starting point at low values of X.

The difference between the present results and the exact solutions is everywhere less than 2 per cent*; this can be seen from Table 1 where the results of the present computations are given with the exact values from reference [2]. Thus, by use of two integral equations it has been possible to attain good accuracy with much less computing time than was needed for the exact solutions.

4. CONCLUSIONS

(i) An approximate method for the solution of the partial differential equation for heat transfer in a uniform-property universal turbulent boundary layer has been described. The method is of profile type; two integral equations are used to obtain the two free parameters in the temperature profile.

(ii) Approximate solutions obtained by use of this method have been compared with available exact solutions. The agreement is very good.

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* In the case of $N_{Pr}/N_{Pr,t} = 30$, the present results differ from the exact values by about 4·5 per cent for large values of X. This is surprising in view of the good agreement for all the other values of $N_{Pr}/N_{Pr,t}$. The reason may be that the exact values from reference [2] are somewhat in error in this case. This conclusion is further supported by the comparison of Table 1 and Table 2 in reference [3]; for large values of X, the value of $(\xi - u^+)$ from Table 1 should be equal to $[(c_f/2)^{1/2}/N_{St} - 1/S_{T,1}]$ from Table 2. It can be seen that this condition is not satisfied with sufficient accuracy for only $N_{Pr}/N_{Pr,t} = 30$.

Table 1. Comparison of values of S_T from present computations with exact values from reference [2]

X	$N_{Pr}/N_{Pr,t} = 0{\cdot}71$ S_T present method	S_T exact	$N_{Pr}/N_{Pr,t} = 1{\cdot}0$ S_T present method	S_T exact	$N_{Pr}/N_{Pr,t} = 7{\cdot}0$ S_T present method	S_T exact	$N_{Pr}/N_{Pr,t} = 30{\cdot}0$ S_T present method	S_T exact	$N_{Pr}/N_{Pr,t} = 100{\cdot}0$ S_T present method	S_T exact	$N_{Pr}/N_{Pr,t} = 1000{\cdot}0$ S_T present method	S_T exact
1×10	0·2250	0·2228	0·2522	0·2498	0·4822	0·4779	0·7831	0·7763	1·1698	1·1598	2·5197	2·4984
2×10	0·1789	0·1771	0·2005	0·1987	0·3830	0·3793	0·6218	0·6158	0·9286	0·9203	1·9999	1·9807
3×10	0·1567	0·1554	0·1755	0·1742	0·3350	0·3321	0·5435	0·5387	0·8115	0·8047	1·7473	1·7317
4×10	0·1428	0·1418	0·1599	0·1589	0·3047	0·3023	0·4942	0·4901	0·7376	0·7317	1·5878	1·5742
5×10	0·1330	0·1322	0·1489	0·1481	0·2833	0·2813	0·4591	0·4555	0·6851	0·6798	1·4743	1·4620
6×10	0·1256	0·1249	0·1406	0·1399	0·2670	0·2653	0·4324	0·4292	0·6450	0·6402	1·3877	1·3764
7×10	0·1198	0·1191	0·1340	0·1335	0·2540	0·2525	0·4111	0·4082	0·6131	0·6086	1·3185	1·3080
8×10	0·1150	0·1144	0·1287	0·1282	0·2434	0·2421	0·3936	0·3910	0·5868	0·5826	1·2641	1·2515
9×10	0·1111	0·1105	0·1242	0·1238	0·2344	0·2334	0·3788	0·3765	0·5645	0·5606	1·2132	1·2038
1×10^2	0·1077	0·1072	0·1204	0·1200	0·2268	0·2259	0·3661	0·3640	0·5454	0·5418	1·1717	1·1628
2×10^2	0·0891	0·0886	0·0997	0·0993	0·1845	0·1844	0·2943	0·2934	0·4363	0·4343	0·9332	0·9260
3×10^2	0·0806	0·0801	0·0906	0·0901	0·1662	0·1665	0·2613	0·2614	0·3849	0·3841	0·8185	0·8135
4×10^2	0·0756	0·0750	0·0851	0·0847	0·1562	0·1564	0·2418	0·2427	0·3536	0·3538	0·7472	0·7435
5×10^2	0·0721	0·0715	0·0814	0·0809	0·1499	0·1499	0·2292	0·2303	0·3323	0·3332	0·6971	0·6946
6×10^2	0·0694	0·0689	0·0786	0·0782	0·1457	0·1455	0·2206	0·2217	0·3168	0·3183	0·6596	0·6581
7×10^2	0·0674	0·0668	0·0765	0·0760	0·1427	0·1423	0·2144	0·2154	0·3053	0·3071	0·6302	0·6295
8×10^2	0·0657	0·0651	0·0747	0·0742	0·1404	0·1398	0·2099	0·2107	0·2964	0·2984	0·6064	0·6064
9×10^2	0·0643	0·0637	0·0732	0·0728	0·1386	0·1379	0·2066	0·2071	0·2895	0·2915	0·5867	0·5874
1×10^3	0·0631	0·0625	0·0720	0·0715	0·1371	0·1363	0·2041	0·2042	0·2840	0·2859	0·5702	0·5714
2×10^3	0·0562	0·0556	0·0649	0·0644	0·1301	0·1288	0·1945	0·1922	0·2634	0·2628	0·4879	0·4910
3×10^3	0·0528	0·0522	0·0615	0·0610	0·1271	0·1258	0·1918	0·1887	0·2593	0·2575	0·4620	0·4638
4×10^3	0·0506	0·0501	0·0592	0·0588	0·1253	0·1240	0·1904	0·1873	0·2577	0·2557	0·4521	0·4527
5×10^3	0·0491	0·0485	0·0577	0·0573	0·1240	0·1227	0·1895	0·1866	0·2569	0·2550	0·4479	0·4478
6×10^3	0·0479	0·0473	0·0564	0·0560	0·1230	0·1217	0·1888	0·1862	0·2564	0·2545	0·4459	0·4455
7×10^3	0·0469	0·0464	0·0554	0·0550	0·1222	0·1209	0·1883	0·1859	0·2560	0·2542	0·4448	0·4444
8×10^3	0·0461	0·0456	0·0546	0·0542	0·1216	0·1202	0·1879	0·1856	0·2557	0·2540	0·4441	0·4438
9×10^3	0·0454	0·0449	0·0539	0·0535	0·1210	0·1196	0·1875	0·1853	0·2554	0·2537	0·4438	0·4434

1×10^4	0·0448	0·0443	0·0532	0·0529	0·1205	0·1191	0·1872	0·1851	0·2552	0·2535	0·4435	0·4431
2×10^4	0·0411	0·0406	0·0494	0·0491	0·1173	0·1152	0·1852	0·1828	0·2539	0·2521	0·4426	0·4420
3×10^4	0·0393	0·0388	0·0474	0·0471	0·1156	0·1129	0·1841	0·1812	0·2532	0·2513	0·4422	0·4418
4×10^4	0·0380	0·0376	0·0461	0·0458	0·1144	0·1113	0·1834	0·1800	0·2527	0·2509	0·4420	0·4418
5×10^4	0·0371	0·0367	0·0451	0·0448	0·1134	0·1102	0·1828	0·1792	0·2524	0·2506	0·4418	0·4418
6×10^4	0·0364	0·0359	0·0444	0·0441	0·1127	0·1093	0·1824	0·1786	0·2521	0·2504	0·4417	0·4417
7×10^4	0·0358	0·0354	0·0437	0·0435	0·1121	0·1087	0·1820	0·1781	0·2519	0·2502	0·4416	0·4417
8×10^4	0·0353	0·0349	0·0432	0·0429	0·1116	0·1082	0·1816	0·1778	0·2517	0·2501	0·4416	0·4417
9×10^4	0·0349	0·0344	0·0427	0·0425	0·1112	0·1078	0·1814	0·1776	0·2515	0·2500	0·4415	0·4417
1×10^5	0·0345	0·0341	0·0423	0·0421	0·1108	0·1075	0·1811	0·1774	0·2514	0·2499	0·4415	0·4416
2×10^5	0·0322	0·0317	0·0398	0·0394	0·1082	0·1060	0·1795	0·1694	0·2504	0·2490	0·4412	0·4414
3×10^5	0·0310	0·0305	0·0384	0·0381	0·1067	0·1052	0·1785	0·1692	0·2499	0·2484	0·4410	0·4412
4×10^5	0·0301	0·0297	0·0375	0·0372	0·1057	0·1044	0·1778	0·1691	0·2495	0·2479	0·4409	0·4411
5×10^5	0·0295	0·0291	0·0369	0·0365	0·1049	0·1037	0·1773	0·1689	0·2491	0·2475	0·4408	0·4410
6×10^5	0·0291	0·0286	0·0363	0·0360	0·1043	0·1031	0·1769	0·1688	0·2489	0·2472	0·4407	0·4409
7×10^5	0·0287	0·0282	0·0359	0·0355	0·1037	0·1025	0·1765	0·1687	0·2487	0·2469	0·4406	0·4408
8×10^5	0·0283	0·0278	0·0355	0·0352	0·1033	0·1019	0·1762	0·1685	0·2485	0·2467	0·4406	0·4407
9×10^5	0·0280	0·0276	0·0352	0·0348	0·1029	0·1013	0·1759	0·1684	0·2483	0·2466	0·4405	0·4407
1×10^6	0·0278	0·0273	0·0349	0·0346	0·1025	0·1008	0·1757	0·1683	0·2482	0·2464	0·4405	0·4407

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